Topologically massive gravity as a Pais-Uhlenbeck oscillator

Özgür Sarıoğlu* and Bayram Tekin†

Department of Physics, Faculty of Arts and Sciences,
Middle East Technical University, 06531, Ankara, Turkey

(Dated: November 7, 2006)

We give a detailed account of the free field spectrum and the Newtonian limit of the linearized "massive" (Pauli-Fierz), "topologically massive" (Einstein-Hilbert-Chern-Simons) gravity in 2+1 dimensions about a Minkowski spacetime. For a certain ratio of the parameters, the linearized free theory is Jordan-diagonalizable and reduces to a degenerate "Pais-Uhlenbeck" oscillator which, despite being a higher derivative theory, is ghost-free.

PACS numbers: 04.60.-m, 03.70.+k, 11.15.-q

I. INTRODUCTION

As is well known, Einstein's General Relativity (GR) in 3+1 dimensions is nonrenormalizable and breaks down at "high" energies. If GR is considered as a low energy effective limit of some unknown fundamental theory, then it will receive higher curvature corrections, such as $\alpha R^2 + \beta R_{\mu\nu}^2$, etc. which give a much well-behaved gravity theory in the UV region, but unfortunately ghosts are not decoupled and unitarity is lost. In fact, in the Einstein plus quadratic curvature model whose Newtonian potential reads

$$V(r) = GM\Big\{ -\frac{1}{r} - \frac{1}{3}\frac{e^{-m_1 r}}{r} + \frac{4}{3}\frac{e^{-m_2 r}}{r} \Big\},$$

one can see the nice UV behavior ($V(0) = 0$) and the presence of the ghost related to the last repulsive term (see [1] and the early references therein). It is, therefore, of extreme importance to get a higher derivative gravity theory without ghost problems. 2+1 dimensional gravity, which is quite easy compared to GR, is a nice laboratory to study various quantum gravity issues. Admittedly, the lesson one learns for 3+1 dimensions should always be taken with a grain of salt. Nevertheless, valuable insight on higher derivative gravity theories, as well as massive gravity theories, can be gained with the study of 2+1 dimensional gravity models. In fact, an analogous approach in quantum field theory has borne much fruit. In this paper, we shall find a particular ghost-free higher derivative gravity model, which has various "mass" terms.

In $2+1$ dimensions, in addition to the usual gauge non-invariant Proca/Pauli-Fierz mass terms, one can add a gauge invariant "mass" which is the Chern-Simons term [2] to both (the abelian and the non-abelian) spin-1 and spin-2 models. [To be more precise, the Chern-Simons term acts like a mass term only at the quadratic level.] There are major differences between the vector and the tensor cases, one of which is related to the gauge invariance: In

*Electronic address: `sarioglu@metu.edu.tr`

†Electronic address: `btekin@metu.edu.tr`

the former, even the Proca mass can be made gauge invariant by introducing a Higgs field [see e.g. [3] and the references therein]; but in the latter, it is simply not known how to get the Pauli-Fierz mass from a gauge (diffeomorphism) invariant action.

The van Dam-Veltman-Zakharov ([4, 5]) discontinuity between the massive and massless gravity models registers as another difference between the *vector* and *tensor* theories. [Recall that the vDVZ discontinuity states that around the Minkowski background, the massless limit of the massive gravity (Einstein-Pauli-Fierz theory) is not the massless gravity (Einstein theory).] The existence of such a discontinuity [15] and perhaps the isolation of massless gravity from the massive (albeit arbitrarily small) ones, is of extreme importance as far as the concept of mass in gravity is concerned. The discontinuity also has much to do with the difference in the degrees of freedom between the massive and massless models.

In a previous Letter [7], the 3-term model (Einstein-Hilbert, Chern-Simons and Pauli-Fierz) was studied in the context of various discontinuities and parameter ranges. The degrees of freedom were identified but the diagonalization of the model (or the explicit form of the free field Lagrangian) was not carried out. As we shall see in this current work, a detailed analysis of the free theory is not redundant at all, since the theory exhibits rather interesting features: For example, as opposed to being a collection of simple harmonic oscillators, we shall get higher order *degenerate* (i.e. equal frequency, mass) Pais-Uhlenbeck oscillators [8] and the model (for a certain ratio of the Chern-Simons and the Pauli-Fierz parameters) can be brought into a Jordan-diagonal form at most. This is rather unexpected since, without the Pauli-Fierz mass term, the free field limit of the model is *not* a higher derivative model. Thus we have a very interesting situation here: the higher derivative model is ghost-free! In fact, in the recent literature, one can find examples of similar ghost-free higher derivative models: Mannheim-Davidson [9], Smilga [10, 11] and Hawking-Hertog [12] all deal with the one-dimensional degenerate Pais-Uhlenbeck oscillator as a model of higher derivative ghost-free theories.

There are a number of reasons why one would be interested in higher derivative models: First of all, they are much better behaved in the UV region (see e.g. Stelle's [13] work in higher curvature gravity) and secondly, even if one does not have higher order terms at the classical level, quantum corrections usually introduce such terms. Ironically, in most of the renormalizable theories, during the process of regularization and renormalization, one introduces ghosts only to remove them at the end! Thus a higher derivative model without ghosts, as the one considered here, would be of some interest.

The layout of the paper is as follows: In section 2, we introduce the 3-term model and work out the free field Lagrangian. In section 3, we suppress the spatial dependencies to basically look at the model as a collection of oscillators. In the first appendix, we give a basic review of the Pais-Uhlenbeck oscillator and the second appendix contains the Newtonian limit of the 3-term model.

II. THE MODEL

Since the signs and the values (or the ratios) of the involved parameters are of extreme importance, here we carefully define and work out the linear version of the 3-term gravitational action which is the sum of Einstein-Hilbert, (third derivative order) Chern-Simons and standard Pauli-Fierz mass, terms

$$I = \int_M d^3x \left\{ a\sqrt{-g}R - \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho}\right) - \frac{m^2}{4}\sqrt{-g}(h_{\mu\nu}h^{\mu\nu} - h^2)\right\}, \tag{1}$$

only at quadratic order in deviations $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ about the flat Minkowski background $\eta_{\mu\nu}$ with the usual definition $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. Our signature is $(-,+,+)$, we use $\epsilon_{012} \equiv 1$ throughout and, in what follows, all operations are carried out with respect to the flat background $\eta_{\mu\nu}$. The model under investigation is the most general such model [7] since the sign of μ is arbitrary, that of m^2 is free and the presence of a allows for choosing the Einstein term's sign ($a = +1$ in the usual case, $a = -1$ for topologically massive gravity (TMG) [2] and $a = 0$ for CS-Pauli-Fierz theory). Note that $\mu \to \infty$ represents massive gravity with 2 excitations and $m = 0$ is TMG with a single mode [2], whereas pure Einstein theory, $1/\mu = m^2 = 0$, has no excitations in $D = 3$. Without working the details, a cursory look at the model does not really reveal what kinds of "oscillators" we have for the free field limit. In fact, as will be seen shortly, there are various discontinuities as far as the limits of the parameters are concerned.

We shall be interested only with the free field limit for now. [Of course, one eventually has to deal with the interacting theory, but, unfortunately, there are infinitely many terms in the interaction Lagrangian rendering the perturbation theory intractable.] The individual terms that make up the action (1) can each be expanded at quadratic order in $h_{\mu\nu}$ as follows: The Einstein-Hilbert piece reads

$$I_E = a\int d^3x\,\sqrt{-g}R = -\frac{a}{2}\int d^3x\, h_{\mu\nu}\,\mathcal{G}_L^{\mu\nu} + O(h^3)\,, \tag{2}$$

where

$$\mathcal{G}_L^{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial^\mu h^{\nu\sigma} + \partial_\sigma\partial^\nu h^{\mu\sigma} - \Box h^{\mu\nu} - \partial^\mu\partial^\nu h\right) - \frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right)\,. \tag{3}$$

The Chern-Simons bit yields

$$I_{CS} = -\frac{1}{2\mu}\int d^3x\,\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho}\right) = -\frac{1}{2\mu}\int d^3x\,\epsilon_{\mu\alpha\beta}\,\mathcal{G}_L^{\alpha\nu}\,\partial^\mu h^\beta{}_\nu + O(h^3)\,. \tag{4}$$

Finally the Pauli-Fierz mass term, up to $O(h^3)$, gives

$$I_{PF} = -\frac{m^2}{4}\int d^3x\,\sqrt{-g}(h_{\mu\nu}h^{\mu\nu} - h^2) = -\frac{m^2}{4}\int d^3x\,(h_{ij}^2 - 2h_{0i}h_{0i} + h_{00}^2 - h^2)\,. \tag{5}$$

It was shown in [2] that the sign of the Einstein term in pure TMG ($m^2 = 0$) must be $a = -1$, opposite to that in the usual Einstein gravity, in order for the energy to be positive, independent of the sign of μ. On the other hand, the usual massive spin-2 system does have excitations and this forces both the relative and overall signs of the Einstein and Pauli-Fierz mass terms to be in the usual way to avoid the presence of ghosts and tachyons. This indicates an inevitable dilemma in the choice of the sign of a in the TMG and massive spin-2 cases.

To analyze the full system given in (1) for generic values of (μ, m) and its dependence on the sign of a, we first decompose $h_{\mu\nu}$ as

$$h_{ij} \equiv (\delta_{ij}+\hat{\partial}_i\hat{\partial}_j)\phi - \hat{\partial}_i\hat{\partial}_j\chi + (\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i)\xi\,, \qquad h_{0i} \equiv -\epsilon_{ij}\partial_j\eta + \partial_i N_L\,, \qquad h_{00} \equiv N\,, \tag{6}$$

where $\hat{\partial}_i \equiv \partial_i/\sqrt{-\partial_k^2}$. The decomposition (6) yields the following for the Einstein (2), Chern-Simons (4) and Pauli-Fierz (5) components of (1), respectively:

$$\begin{aligned} I_E &= \frac{a}{2}\int d^3x \left\{\phi\ddot{\chi} + \phi\nabla^2(N - 2\dot{N}_L) + (-\nabla^2\eta + \dot{\xi})^2\right\} , \qquad (7)\\ I_{CS} &= \frac{1}{2\mu}\int d^3x \left\{(-\nabla^2\eta + \dot{\xi})[\nabla^2(N - 2\dot{N}_L) + \ddot{\chi} + \Box\phi]\right\} , \qquad (8)\\ I_{PF} &= -\frac{m^2}{2}\int d^3x \left\{N_L\nabla^2 N_L + \eta\nabla^2\eta + \xi^2 - \phi\chi + N(\phi + \chi)\right\} . \qquad (9)\end{aligned}$$

When all these are put together, the terms proportional to N can be collected altogether to obtain

$$\begin{aligned} I &= \frac{1}{2}\int d^3x \left\{a(\phi\ddot{\chi} - 2\phi\nabla^2\dot{N}_L + (-\nabla^2\eta + \dot{\xi})^2) + \frac{1}{\mu}(-\nabla^2\eta + \dot{\xi})[-2\nabla^2\dot{N}_L + \ddot{\chi} + \Box\phi]\right.\\ &\left. -m^2[N_L\nabla^2 N_L + \eta\nabla^2\eta + \xi^2 - \phi\chi] + N\left[a\nabla^2\phi - \frac{1}{\mu}\nabla^2(\nabla^2\eta - \dot{\xi}) - m^2(\phi + \chi)\right]\right\} . \qquad (10)\end{aligned}$$

This looks like a highly complicated system. To simplify it, the next thing to do is to use integrations by parts, whenever necessary, to eliminate those variables from (10) which can be treated as 'Lagrange multipliers' and to use the corresponding 'constraint equations' thus obtained so that only dynamical degrees of freedom are left. When N, η and N_L are eliminated in this fashion, one finds

$$I = \frac{1}{2}\int d^3x \left[a\phi\Box\phi - m^2\phi^2 + 2\lambda\Box\phi + a\mu^2\lambda^2 - m^2(\mu\lambda - \dot{\xi})\frac{1}{\nabla^2}(\mu\lambda - \dot{\xi}) - m^2\xi^2\right] , \qquad (11)$$

where we have used $\lambda \equiv -a\phi + m^2\frac{1}{\nabla^2}(\phi + \chi)$ for simplicity. The remaining three dynamical components of $h_{\mu\nu}$ yield

$$\begin{aligned} \delta\phi &: \qquad \Box\lambda + a\Box\phi - m^2\phi = 0 , \qquad (12)\\ \delta\lambda &: \qquad \Box\phi - m^2\mu^2\frac{1}{\nabla^2}\lambda + m^2\mu\frac{1}{\nabla^2}\dot{\xi} + a\mu^2\lambda = 0 , \qquad (13)\\ \delta\xi &: \qquad \mu\dot{\lambda} + \Box\xi = 0 , \qquad (14)\end{aligned}$$

which together give

$$(\Box^3 - a^2\mu^2\Box^2 + 2am^2\mu^2\Box - m^4\mu^2)\phi = 0 , \qquad (15)$$

for the field ϕ, through which the remaining fields are determined as

$$\lambda = \left(-a + m^2\frac{1}{\Box}\right)\phi \qquad \text{and} \qquad \xi = \frac{1}{\Box}\left(a\mu - m^2\mu\frac{1}{\Box}\right)\dot{\phi} .$$

Equation (15) was obtained in [7] and [14], in which it was shown that for $a = 1$, the roots of the eigenvalue equation are complex unless $\mu^2/m^2 \geq 27/4$.

Introducing auxiliary variables $\psi \equiv \Box\phi$ and $\Omega \equiv \Box^2\phi = \Box\psi$, the equation for ϕ (15) can be put in the form $\Box\mathbf{x} = \mathbf{A}\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix}\phi\\ \psi\\ \Omega\end{bmatrix} , \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ m^4\mu^2 & -2am^2\mu^2 & a^2\mu^2\end{bmatrix} .$$

In passing, we should note that when $\mu^2/m^2 > 27/4$, the eigenvalue equation has three distinct real roots, the explicit forms of which are rather cumbersome so that we refrain from presenting them here. However, the free field limit of the model can simply be thought of as being described by three uncoupled oscillators then.

The degenerate case:

Let us now consider the case when $a = 1$ and $\mu^2/m^2 = 27/4$. Then $\det(p\mathbf{I} - \mathbf{A}) = (p - 3m^2)^2 \, (p - \frac{3}{4}m^2)$ and by using the eigenvectors of $\mathbf{A}$, one can form the modal matrix of $\mathbf{A}$ as

$$\mathbf{P} = \begin{bmatrix} 16/9 & 1 & 0 \\ 4m^2/3 & 3m^2 & 1 \\ m^4 & 9m^4 & 6m^2 \end{bmatrix},$$

which 'Jordan-diagonalizes' $\mathbf{A}$ as

$$\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 3m^2/4 & 0 & 0 \\ 0 & 3m^2 & 1 \\ 0 & 0 & 3m^2 \end{bmatrix}.$$

By introducing $\mathbf{z} = \mathbf{P}^{-1}\mathbf{x} = [z_1 \; z_2 \; z_3]^T$, this system thus takes the form $\Box\mathbf{z} = \mathbf{\Lambda}\mathbf{z}$, which can be thought of as coming from an equivalent dynamical system whose action reads

$$I_{\text{eq}} = \frac{1}{2}\int d^3x \left[z_1 \left(\Box - \frac{3}{4}m^2 \right) z_1 + z_2(\Box - 3m^2)^2 z_2 \right],$$

where specifically

$$z_1 = \left(1 - \frac{2}{3m^2}\Box + \frac{1}{9m^4}\Box^2 \right)\phi \quad \text{and} \quad z_2 = \left(-\frac{7}{9} + \frac{32}{27m^2}\Box - \frac{16}{81m^2}\Box^2 \right)\phi\,.$$

The first term represents the usual (ghost-free) massive real scalar field whose quantization is well-known. The second term describes a degenerate (or purely quadratic) Pais-Uhlenbeck oscillator. In principle, just like any other higher derivative model, one expects this model to be contaminated with ghosts. However, as was shown in [9, 10, 12], this is not true: Purely quadratic models differ from those with quadratic plus lower derivative ones. This should not be surprising because, even the classical solutions of the purely quadratic theories have milder instabilities and physically acceptable parameter ranges.

This, of course, is all at the level of the free-field theory. When interactions are introduced, the picture will necessarily change drastically. We are not aware of much work on this, save the work of Smilga [11] who has shown that once a quartic interaction is introduced, depending on the numerical values of the ratios of the involved parameters (such as the coupling constant to mass ratio), the model still has ghost-free, stable regions. This issue obviously deserves more attention.

III. THE REDUCED MODELS

In this section, we shall suppress the spatial part of the theory to see more transparently the degrees of freedom (the free oscillators) of the model. The advantage of this is that (10) simplifies a great deal without losing any of its degrees of freedom. Much of what we have done above (in a fully relativistic way) can be understood more easily from the following reduced Lagrangian

$$I = \frac{1}{2}\int d^3x \left\{ a(\phi\ddot{\chi} + \dot{\xi}^2) + \frac{1}{\mu}\dot{\xi}\,(\ddot{\chi} - \ddot{\phi}) - m^2(\xi^2 - \phi\chi + N(\phi + \chi)) \right\} . \tag{16}$$

Getting rid of the Lagrange multiplier N, we find the reduced form of the 3-term model:

$$I = \frac{1}{2}\int d^3x \left\{ a(\dot{\chi}^2 + \dot{\xi}^2) + \frac{2}{\mu}\,\dot{\xi}\,\ddot{\chi} - m^2(\xi^2 + \chi^2) \right\} , \tag{17}$$

which leads to the following equations of motion:

$$\ddot{\chi} + m^2\chi = \frac{1}{\mu}\dddot{\xi} , \qquad \ddot{\xi} + m^2\xi = -\frac{1}{\mu}\dddot{\chi} . \tag{18}$$

Let us now check various limits:

The pure TMG case:

Setting the Pauli-Fierz mass to zero ($m^2 = 0$) and eliminating $\dot{\xi}$, which behaves like a Lagrange multiplier, we have

$$I = \frac{1}{2}\int d^3x \left\{ -a(\dot{\chi}^2 - a^2\mu^2\chi^2) \right\} . \tag{19}$$

It is clear that when $a = -1$, we have a ghost-free, massive, single excitation. This is the good old result of [2].

Einstein-Pauli-Fierz case:

Setting $\mu \to \infty$, we get

$$I = \frac{1}{2}\int d^3x \left\{ a(\dot{\chi}^2 + \dot{\xi}^2) - m^2(\xi^2 + \chi^2) \right\} , \tag{20}$$

which describes two real scalar, massive degrees of freedom if $a = +1$. [This is what one expects from the onset, since massive spin-2 field has 2 degrees of freedom in 2+1 dimensions, that is as much as the massless spin-2 field in 3+1 dimensions.]

No Einstein term:

Setting $a = 0$ yields a higher derivative model

$$I = \frac{1}{2}\int d^3x \left\{ \frac{2}{\mu}\,\dot{\xi}\,\ddot{\chi} - m^2(\xi^2 + \chi^2) \right\} , \tag{21}$$

which has to be diagonalized. However, as the eigenvalue equation ($M^3 + |\mu| m^2 = 0$), where M is the mass of the excitations, constructed from the equations of motion, shows, this model has tachyonic solutions. This could be of some interest in Anti-de Sitter spacetime backgrounds which allows for certain negative mass squared solutions, but is not viable in flat backgrounds.

IV. CONCLUSIONS

In this work, we have studied massive gravity (with two different types of mass: the Chern-Simons and the Pauli-Fierz) in 2+1 dimensional gravity. The theory has interesting mass discontinuities: It becomes a higher order (degenerate) Pais-Uhlenbeck oscillator if the ratio of the Chern-Simons and the Pauli-Fierz masses is tuned to $3\sqrt{3}/2$. The full theory (i.e. the Einstein-Chern-Simons-Pauli-Fierz model) generically has, at the linearized level, 3 distinct massive degrees of freedom. The free field Lagrangian is a collection of these 3 oscillators. Therefore, it was surprising to see that when the parameters are tuned as mentioned above, two of the masses coalesce and one gets a higher derivative oscillator in addition to a normal one. The model at that limit is at most Jordan-diagonalizable. It is a ghost-free higher-derivative model whose properties were put under scrutiny recently [9, 10, 12], where quantization was also carried out.

In this work, we have only considered the non-interacting theory. It would be quite interesting to consider the lowest order interacting theory to see whether the higher derivative model stays ghost-free (or as far as the classical solutions are concerned, whether there are stable regions or not). Such an analysis was carried out by Smilga [11] in a simplified model.

APPENDIX A: THE PAIS-UHLENBECK OSCILLATOR

Here we would like to recapitulate the nonrelativistic limit of the higher derivative field theories considered a long time ago by [8]. Pais and Uhlenbeck studied a generic scalar field theory model whose Lagrangian is of the form

$$\mathcal{L} = -\frac{1}{2}\phi\Big(\prod_{i=1}^{N}(\Box - M_i^2)\Big)\phi\,.$$

Here we will take $N = 2$ and examine only the nonrelativistic limit, which basically means the dropping down of the spatial derivatives. Thus consider a Lagrangian of the form

$$L = \frac{1}{2}\Big\{\ddot{q}^2 - (\omega_1^2 + \omega_2^2)\dot{q}^2 + \omega_1^2\omega_2^2 q^2\Big\}\,, \tag{A1}$$

and note that $\omega_1 \neq \omega_2$ and $\omega_1 = \omega_2$ cases differ drastically. We will call the latter case as the degenerate Pais-Uhlenbeck oscillator which is similar to the case studied in this paper.

The naive Ostrogradski Hamiltonian simply reads

$$H = \frac{\ddot{q}^2}{2} - \dot{q}\,\dddot{q} - \frac{1}{2}(\omega_1^2 + \omega_2^2)\dot{q}^2 - \frac{1}{2}\omega_1^2\omega_2^2 q^2\,, \tag{A2}$$

which gives the following, not necessarily positive, energies for solutions of the equation of motion

$$E = \begin{cases} \frac{1}{2}(\omega_1^2 - \omega_2^2)\Big[\omega_1^2(c_1^2 + c_2^2) - \omega_2^2(c_3^2 + c_4^2)\Big], & \omega_1 \neq \omega_2 \\ 2\omega^2\Big[\omega(c_5 c_6 - c_7 c_8) + c_5^2 + c_8^2\Big], & \omega_1 = \omega_2 = \omega \end{cases}\,, \tag{A3}$$

where $c_i (i = 1, \ldots, 8)$ are constants. To be able to quantize these theories, a proper Hamiltonian in terms of momenta can be defined by using Dirac's constraint analysis. Taking $\dot{q} = \pi$, one gets

$$H = \frac{p_\pi^2}{2} + \frac{1}{2}(\omega_1^2 + \omega_2^2)\pi^2 - \frac{1}{2}\omega_1^2\omega_2^2 q^2 + p_q\pi\,, \tag{A4}$$

where p_π and p_q are the usual canonical momenta. One can see the glimpses of two coupled oscillators here, but one still needs to "diagonalize" this Hamiltonian. For $\omega_1 = \omega_2$, just like the oscillator studied in the bulk of this paper, the Hamiltonian is only Jordan diagonalizable. For $\omega_1 \neq \omega_2$, one can use the following Pais-Uhlenbeck variables

$$Q_1 \equiv q + \frac{\ddot{q}}{\omega_2^2}\,, \qquad Q_2 \equiv q + \frac{\ddot{q}}{\omega_1^2}\,. \tag{A5}$$

In this case the Hamiltonian becomes

$$H = \frac{1}{2}\frac{\omega_1^4}{\omega_2^2 - \omega_1^2}(\dot{Q}_2^2 + \omega_2^2 Q_2^2) - \frac{1}{2}\frac{\omega_2^4}{\omega_2^2 - \omega_1^2}(\dot{Q}_1^2 + \omega_1^2 Q_1^2)\,, \tag{A6}$$

which is a collection of two oscillators except that one has the wrong sign. For a discussion of the quantization of the $\omega_1 \neq \omega_2$ case, we refer the reader to [9, 10].

APPENDIX B: THE SOLUTIONS

Let us now examine the steady-state solutions of the $D = 3$ Pais-Uhlenbeck oscillator. For this purpose, we assume that the mass parameter m has been scaled appropriately and the field ϕ (or z_2 thereof) has been redefined accordingly such that the problem reduces to investigating the solutions of the $(\nabla^2 - m^2)^2\varphi = 0$ equation on the (2-spatial dimensional full) plane. Below we will give the exact solution of this equation in terms of highly complicated special functions. However, one can easily extract the asymptotic behavior of the solution at spatial infinity, which is simply $\varphi \sim (c_1 + c_2 r)e^{-mr}$. Of course, we have assumed "nice" boundary conditions and discarded the divergent parts to find this.

First consider the simpler equation $(\nabla^2 - m^2)\varphi = 0$. For the problem at hand, one can assume circular symmetry, take $\varphi = \varphi(r,\theta)$, employ a 'separation of variables' ansatz, introduce a new independent variable $x \equiv mr$, arrive at the well known modified Bessel equation and, further assuming the solutions to be free of any angular dependency, write down

$$\varphi(r,\theta) = R(mr) \equiv R(x) = \alpha\, I_0(x) + \beta\, K_0(x)\,,$$

for some integration constants α and β. Thus, our original problem reduces to examining the solutions of the following ordinary differential equation

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - R(x) = \beta\, K_0(x)\,,$$

if we further demand that the solutions of $(\nabla^2 - m^2)^2\varphi = 0$ are finite and regular at large x (or r). One can view this as a two-point boundary condition problem, where the solution $R(x)$ is required to be finite and bounded for both small and large values of x. Using standard Green's functions' techniques, one arrives at

$$R(x) = -\beta\left(\int_0^x K_0(x)\, I_0(y)\, K_0(y)\, y\, dy + \int_x^\infty I_0(x)\, K_0^2(y)\, y\, dy\right)\,,$$

which can be evaluated using the software package `Mathematica`. The outcome of this is

$$R(x) = \varphi(mr) = -\beta\left(\frac{1}{2}x^2\, I_0(x)(-K_0^2(x) + K_1^2(x)) + \frac{1}{4\sqrt{\pi}}\, K_0(x)\, u(x)\right)\,,$$

where

$$u(x) = \text{MeijerG}[\{\{3/2\},\{\ \}\},\{\{1,1\},\{0\}\}\},x^2] = \frac{1}{2\pi i}\int \frac{\Gamma(-s-1/2)\,(\Gamma(s+1))^2}{\Gamma(1-s)}\,x^{-2s}\,ds\,,$$

in which the contour of integration on the s-plane is between the poles of $\Gamma(-s-1/2)$ and $\Gamma(s+1)$. By using the properties of the Γ-functions, one can also write

$$u(x) = -\frac{1}{2\pi i}\int \frac{s^2\,\Gamma(s)\,\tan\pi s}{\Gamma(s+3/2)}\,x^{-2s}ds\,,$$

and note that $\Gamma(-s-1/2)$ has poles at $s=-1/2,1/2,3/2,\ldots$ and $\Gamma(s+1)$ has poles at $s=-1,-2,-3,\ldots$.

ACKNOWLEDGMENTS

This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK). B.T. is also partially supported by the Turkish Academy of Sciences (TÜBA) and by the TÜBİTAK Kariyer Grant No. 104T177.

[1] H. Von Borzeszkowski, H. J. Treder and W. Yourgrau, Annalen Phys. **35** (1978) 471.
[2] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. **140**, 372 (1982); Ann. Phys. **185**, 406(E) (1988).
[3] P.N. Tan, B. Tekin and Y. Hosotani, Nucl. Phys. B **502**, 483 (1997), [arXiv:hep-th/9703121].
[4] H. van Dam and M.J.G. Veltman, Nucl. Phys. B **22**, 397 (1970).
[5] V.I. Zakharov, JETP Lett. **12** 312 (1970); [Pisma Zh. Eksp. Teor. Fiz. **12** 447 (1970)].
[6] A.I. Vainshtein, Phys. Lett. B **39** 393 (1972).
[7] S. Deser and B. Tekin, Class. Quant. Grav. **19**, L97 (2002).
[8] A. Pais and G.E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
[9] P.D. Mannheim and A. Davidson, Phys. Rev. A **71**, 042110 (2005), [arXiv:hep-th/0408104].
[10] A.V. Smilga, Phys. Lett. B **632**, 433 (2006), [arXiv:hep-th/0503213].
[11] A.V. Smilga, Nucl. Phys. B **706**, 598 (2005), [arXiv:hep-th/0407231].
[12] S.W. Hawking and T. Hertog, Phys. Rev. D **65**, 103515 (2002), [arXiv:hep-th/0107088].
[13] K.S. Stelle, Gen. Rel. Grav. **9**, 353 (1978).
[14] C. Pinheiro, G.O. Pires and N. Tomimura, Nuovo Cim. B **111**, 1023 (1996), [arXiv:gr-qc/9704004].
[15] According to Vainshtein [6], the discontinuity should disappear once non-perturbative corrections are taken into account: Namely, in addition to the mass of the graviton, dimensionful scales, related to the Schwarzschild radius of, say, one of the scattering particles should become relevant. This is a plausible idea since only then, the small graviton mass limit seems to make sense, however we are not aware of a full demonstration of Vainshtein's conjecture.